COMPARABILITY OF CLOPEN SETS IN A ZERO-DIMENSIONAL DYNAMICAL SYSTEM

HISATOSHI YUASA

Abstract. Let φ be a homeomorphism on a totally disconnected, compact metric space X. Then, the following are equivalent:

(i) φ is uniquely ergodic;
(ii) any clopen subsets of X are comparable with respect to a certain binary relation;
(iii) an ordered group, which is a quotient of the group of integer-valued continuous functions modulo infinitesimals, is totally ordered.

1. Introduction

Let φ be a homeomorphism on a totally disconnected, compact metric space X. Let M_φ denote the set of φ-invariant probability measures. For clopen sets $A, B \subset X$, we write $A \geq B$ either if $\mu(A) > \mu(B)$ for all $\mu \in M_\varphi$, or if $\mu(A) = \mu(B)$ for all $\mu \in M_\varphi$. This relation $\geq$ does not necessarily hold between given clopen sets (Remark 2.4). If φ is minimal, then $A \geq B$ induces an embedding of B into A via finite or countable Hopf-equivalence [8]. The embedding plays significant roles in analyses of orbit structures of Cantor minimal systems [7, 8, 9] and also in those for locally compact Cantor minimal systems [11]. We refer the reader to [12, 13] for other facts concerning Hopf-equivalence.

Another important object in analyses of the orbit structures is ordered group. Let G_φ denote the quotient group of the abelian group $C(X, \mathbb{Z})$ of integer-valued continuous functions on X by a subgroup:

$$Z_\varphi = \{f \in C(X,\mathbb{Z}) | \int_X f d\mu = 0 \text{ for all } \mu \in M_\varphi\}.$$

Let $G_\varphi^+ = \{[f] \in G_\varphi | f \geq 0\}$, where $[f]$ is the equivalence class of $f \in C(X, \mathbb{Z})$. If φ is minimal, then the ordered group (G_φ, G_φ^+) with the canonical order unit is a complete invariant for orbit equivalence [6].

If φ is uniquely ergodic, then any clopen subsets of X are comparable. This fact may lead us to have a question whether a non-uniquely ergodic system always has incomparable clopen sets, or not. The goal of this paper is to give an affirmative answer to this question:

Theorem 1.1. *The following are equivalent*:

(i) *φ is uniquely ergodic*;
(ii) *any two clopen subsets of X are comparable*;
(iii) *the ordered group (G_φ, G_φ^+) is totally ordered.*

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2. Preliminaries

We freely use terminology concerning (partially) ordered groups or dimension groups; see [3, 4, 6] for example. Let $K^0(X,\varphi)$ denote a quotient group $C(X,\mathbb{Z})/B_\varphi$, where $B_\varphi = \{f\circ\varphi - f | f \in C(X,\mathbb{Z})\}$. Put $K^0(X,\varphi)^+ = \{[f] \in K^0(X,\varphi) | f \geq 0\}$. If any point in X is chain recurrent for φ, then $(K^0(X,\varphi), K^0(X,\varphi)^+)$ becomes an ordered group [1]. This fact is proved also by [13] in connection with finite Hopf-equivalence. If φ is minimal (resp. almost minimal), then $(K^0(X,\varphi), K^0(X,\varphi)^+)$ becomes a simple (resp. almost simple) dimension group [10] (resp. [2]). In each of these cases, $(K^0(X,\varphi), K^0(X,\varphi)^+)$ with the canonical order unit $[\chi_X]$ is a complete invariant for strong orbit equivalence [6, 2], where χ_X is the characteristic function of X.

Suppose φ has a unique minimal set. By [10, Theorem 1.1], any point is chain recurrent for φ. Given $\mu \in M_\varphi$, define a state τ_μ on $(K^0(X,\varphi), [\chi_X])$ by for $f \in C(X,\mathbb{Z})$,

$$\tau_\mu([f]) = \int_X f d\mu.$$

The map $\mu \mapsto \tau_\mu$ is a bijection between M_φ and the set of states on $(K^0(X,\varphi), [\chi_X])$; see for details [10, Theorem 5.5].

Proposition 2.1. (G_φ, G_φ^+) *is an ordered group.*

Proof. Suppose $[f] \in G_\varphi^+ \cap (-G_\varphi^+)$ with $f \in C(X,\mathbb{Z})$. There are nonnegative $g_1, g_2 \in C(X,\mathbb{Z})$ such that $f - g_1, f + g_2 \in Z_\varphi$. Since for all $\mu \in M_\varphi$,

$$0 = \int_X (g_1+g_2)d\mu \geq \int_X g_1 d\mu \geq 0,$$

we obtain $[f] = [g_1] = 0$, i.e. $G_\varphi^+ \cap (-G_\varphi^+) = \{0\}$. Other requirements for (G_φ, G_φ^+) to be an ordered group are readily verified. □

Definition 2.2. Clopen subsets A and B of X are said to be *countably Hopf-equivalent* if there exist $\{n_i \in \mathbb{Z} | i \in \mathbb{Z}^+\}$ and disjoint unions

$$A = \bigcup_{i\in\mathbb{N}} A_i \cup \{x_0\} \text{ and } B = \bigcup_{i\in\mathbb{N}} B_i \cup \{y_0\}$$

into nonempty clopen sets A_i, B_i and singletons $\{x_0\}, \{y_0\}$ such that

(1) $\varphi^{n_0}(x_0) = y_0$ and $\varphi^{n_i}(A_i) = B_i$ for every $i \in \mathbb{N}$;
(2) the map $\alpha : A \to B$ defined by

$$\alpha(x) = \begin{cases} \varphi^{n_i}(x) & \text{if } x \in A_i \text{ and } i \in \mathbb{N}; \\ y_0 & \text{if } x = x_0 \end{cases}$$

is a homeomorphism.

We shall refer to α as a *countable equivalence map* from A onto B.

Lemma 2.3. *Suppose φ is minimal. Let $A, B \subset X$ be clopen. Then, the following are equivalent*:

(1) $A \geq B$;
(2) *there is a countable equivalence map from* B *into* A;
(3) $[\chi_A] - [\chi_B] \in D_\varphi := \{[\chi_C] \in G_\varphi | C \subset X$ *is clopen.*$\}$.

Proof. By [8, Proposition 2.6], (1) is equivalent to (2). If $\alpha : B \to \alpha(B) \subset A$ is a countable equivalence map, then

$$[\chi_A] - [\chi_B] = [\chi_A] - [\chi_{\alpha(B)}] = [\chi_{A\setminus\alpha(B)}] \in D_\varphi.$$

Hence, (2) implies (3). If $[\chi_A] - [\chi_B] = [\chi_C]$ for some clopen set $C \subset X$, then $\int(\chi_A - \chi_B)d\mu = \mu(C) \geq 0$ for all $\mu \in M_\varphi$. Since given a clopen set $C \subset X$, either $\mu(C) = 0$ for all $\mu \in M_\varphi$, or $\mu(C) > 0$ for all $\mu \in M_\varphi$, (3) implies (1). This completes the proof. □

Remark 2.4. One can find a homeomorphism having incomparable clopen sets. Let (X, φ) be a Cantor minimal system such that $K^0(X, \varphi)$ is order isomorphic to $\mathbb{Q}^2$ with the strict ordering by an isomorphism ι mapping the canonical order unit $[\chi_X]$ to $(1, 1)$; see for details [6, 3, 10]. The homeomorphism φ has exactly two ergodic probability measures corresponding to states $\tau_i : \mathbb{Q}^2 \to \mathbb{Q}$ $(i = 1, 2)$ which are the projections to the i-th coordinate. By [8, Lemma 2.4], there exist clopen sets $C, D \subset X$ such that $\iota([\chi_C]) = (1/2, 1/3)$ and $\iota([\chi_D]) = (1/2, 2/3)$, which are incomparable.

3. A proof of the theorem

(ii) ⇒ (iii): We first show that φ has a unique minimal set on which any $\mu \in M_\varphi$ is supported. Let Y be a minimal set. Suppose $\mu \in M_\varphi$ is supported on Y. Suppose $\nu \in M_\varphi$ is different from μ. Assume $\nu(A) > 0$ for a clopen set $A \subset X \setminus Y$. Define $\nu' \in M_\varphi$ by $\nu'(U) = \nu(U \setminus Y)/\nu(X \setminus Y)$ for a measurable set U. By regularity, there exists a clopen set B containing Y such that $\nu'(B) < \nu'(A)$. However, $\mu(B) = 1$ and $\mu(A) = 0$. This contradicts (ii) .

In the remainder of this proof, we tacitly use Lemma 2.3. The fact proved in the preceding paragraph allows us to assume the minimality of φ. Given $a \in G_\varphi$, choose $\{a_i, b_j \in D_\varphi \setminus \{0\} | 1 \leq i \leq n, 1 \leq j \leq m\}$ so that $a = a_1 + a_2 + \cdots + a_n - b_1 - b_2 - \cdots - b_m$.

The following procedure consisting of steps determines $a \geq 0$ or $a \leq 0$.
Step 1. If $\sum_{i=1}^n a_i - b_1 \leq 0$, then $a \leq 0$, and the procedure ends. Otherwise, there is k_1 for which $c_{k_1} := \sum_{i=1}^{k_1} a_i - b_1 \in D_\varphi \setminus \{0\}$ and $a = c_{k_1} + a_{k_1+1} + \cdots + a_n - b_2 - b_3 - \cdots - b_m$. By this operation, the number of terms b_i decreases by one. We may write $a = a_{k_1} + a_{k_1+1} + \cdots + a_n - b_2 - b_3 - \cdots - b_m$.
Step 2. If $\sum_{i=k_1}^n a_i - b_2 \leq 0$, then $a \leq 0$, and the procedure ends. Otherwise, there is $k_2 \geq k_1$ for which $c_{k_2} := \sum_{i=k_1}^{k_2} a_i - b_2 \in D_\varphi \setminus \{0\}$ and $a = c_{k_2} + a_{k_2+1} + \cdots + a_n - b_3 - b_4 - \cdots - b_m$. By this operation, the number of terms b_i decreases by one. We may write $a = a_{k_2} + a_{k_2+1} + \cdots + a_n - b_3 - b_4 - \cdots - b_m$.

Now, it is clear how we should execute each step. The procedure necessarily ends by Step m. We obtain $a \geq 0$ exactly when the procedure ends at Step m.

(iii) ⇒ (i): Assume the existence of a clopen set $A \subset X$ such that $c_2 := \inf_{\mu\in M_\varphi} \int \chi_A d\mu < \sup_{\mu\in M_\varphi} \int \chi_A d\mu =: c_1$. Let $\mu_1, \mu_2 \in M_\varphi$ be so that $c_i = \int \chi_A d\mu_i$. Take $m, n \in \mathbb{N}$ so that $c_2 < n/m < c_1$. Then, $\int(m\chi_A - n)d\mu_1 > 0$

and $\int(m\chi_A - n)d\mu_2 < 0$. This contradicts (iii). This completes the proof of the theorem.

Remark 3.1. The proof of (ii) $\Rightarrow$ (iii) is based on an idea implied in [5, Subsection 5.4].

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17-23-203 Idanakano-cho, Nakahara-ku, Kawasaki Kanagawa 211-0034, JAPAN.
E-mail address: hisatoshi_yuasa@ybb.ne.jp